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FORMULATIONS OF THE PROBLEMS OF THE THEORY OF AN IDEALLY ELASTIC-PLASTIC BODY*

G.A. SEREGIN

Various functional formulations are given for the problems of the theory of quasistaic equilibrium of ideally elastic-plastic media. The first formulation (problem A) follows naturally form the classical formulation. The set of kinematically admissible fields corresponding to it has maximum permissible thickness under the assumption that the deformation rate tensor is summable It is shown that problem A is equivalent to two partial problems (problem B and C). Problem B represents an evolutionary variational inequality for the stress tensor, which has a unique solution. In problem C the known stress field is used to determine the velocity field as a solution of some variational problem depending on the load parameter. It is shown that problem C, and hence problem A, may have no solutions. A variational extension of problem C (problem C^+) is constructed. Problems B and C^+ lead to an enlarged formulation of the classical problem (problem A^+). It is shown that A^+ always has a solution. An example is given, in which A has no solution and A^+ has a unique solution.

Problems concerning the mathematical correctness of the problem of ideal plasticity have been studied by many authors (see e.g. /1—12/. The approach proposed below removes a number of the restrictions in /4—11/.

Problem C with fixed load parameter resembles, in the mathematical sense, the variational problem of deformation plasticity which has been intensely studied in the last few years (see e.g. /7—9/. In /7, 9/ the problem was extended to the space of displacements for which the deformation tensor is a Radon measure. The necessary extremal conditions are expressed here in terms of functions of measures (7, 9/. In the present paper the extension is produced by another method which makes it possible to obtain the relations connecting the velocity and stress fields sought (problem A^+) in terms of point functions only, and this simplifies the solution of specific problems. On the other hand, the definition of admissible sets of problem A^+ implies that the deformation rate tensor is a Radon measure which depends on the load parameter.

1. Direct functional formulation of the classical problem.

Let

$$v = (v_i),\ u = (u_i),\ \tau = (\tau_{ij}),\ \sigma = (\sigma_{ij})\ (i, j = 1, 2, \ldots, n)$$

denote certain vectors and symmetric tensors. We will use the following notation:

$$uv = u_iv_i,\ \sigma\tau = \sigma_{ij}\tau_{ij},\ |u|^2 = uu,\ |\sigma|^2 = \sigma\sigma$$
$$\sigma^D = \sigma - n^{-1}\sigma_{ii}E,\ E = (\delta_{ij})$$

where σ_{ii}, σ^D is the trace of the deviator of the tensor and σ, E is the unit tensor.

The classical initial boundary value problem on the quasistatic equilibrium of a perfectly elastic-plastic body is confirmed to determining the functions u and σ from relations of the form /1, 13/.

$$\begin{aligned}
&\operatorname{div} \sigma(x, t) + f(x, t) = 0,\ |\sigma^D(x, t)| \leqslant \sqrt{2}k_* \\
&\left(\varepsilon(u(x, t)) - \frac{1}{n^2K_0}\sigma_{ii}^{\cdot}(x, t)E - \frac{1}{2\mu}\sigma^{\cdot D}(x, t)\right)(\tau - \sigma(x, t)) \leqslant 0 \\
&\forall\tau: |\tau^D| \leqslant \sqrt{2}k_*,\quad x \in \Omega \\
&\sigma_{ij}(x, t)\nu_j(x) = F_i(x, t),\ x \in \gamma \\
&u(x, t) = U(x, t),\ x \in \Gamma\setminus\gamma;\quad \sigma(x, 0) = \sigma_0(x),\ x \in \Omega \\
&2\varepsilon(u) = (u_{i,j} + u_{j,i}),\ \operatorname{div}\sigma = (\sigma_{ij,j})
\end{aligned} \tag{1.1}$$

Here Ω is a bounded region whose boundary Γ satisfies the Lipshitz condition, γ is the measurable part of Γ, ν is the external normal to Γ, f and F are given loads, U is the known velocity field, a dot denotes differentiation with respect to $t \in [0, T]$, K_0, k_*, μ are positive constants.

We will restrict ourselves to the case of mixed boundary value problem, noting that the study of the first and second boundary value problem does not require any fundamental changes.

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We assume that

$$f, f^{\cdot} \in L^{\infty}(0, T; L^{n}(\Omega)^{n}); \quad F, F^{\cdot} \in L^{\infty}(0, T; L^{\infty}(\gamma)^{n}) \tag{1.2}$$
$$\exists u_0 \in C([0, T]; H^1(\Omega)^n): u_0 = U \text{ on } \Gamma \setminus \gamma$$

where $H^1(\Omega)^n$ is a Sobolev space of vector functions with finite norm

$$|||u||| = \Big(\int_{\Omega} (|u|^2 + u_{i,j}u_{i,j})\,dx\Big)^{1/2}$$

In order to produce a functional formulation of problem (1.1), we shall introduce the following spaces:

$$\Sigma = \{\tau : \|\tau\|_{\Sigma} = \|\tau_{ii}\|_{L^n(\Omega)} + \||\tau^D|\|_{L^{\infty}(\Omega)} < +\infty\}$$
$$D^2(\Omega) = \Big\{v : \|v\|_2 = \int_{\Omega} (|v| + |\varepsilon^D(v)|)\,dx + \|\operatorname{div} v\|_{L^n(\Omega)} < +\infty\Big\}$$

The space $D^2(\Omega)$ imbeds continuously into the spaces $L^{n/(n-1)}(\Omega)^n$ and $L^1(\Gamma)^n$ /2/ of summable functions; therefore we can define auxilliary sets of the form

$$V = \{v \in D^2(\Omega): v = 0 \text{ on } \Gamma \setminus \gamma\}$$
$$D(A^*) = \{(\tau, g) : \tau \in \Sigma, \operatorname{div}\tau \in L^n(\Omega)^n, \quad g \in L^{\infty}(\gamma)^n$$
$$\int_{\Omega} (\tau\varepsilon(v) + v \operatorname{div}\tau)\,dx = \int_{\gamma} gv\,d\Gamma, \ \forall v \in V\Big\}$$
$$Q(t) = \{\tau: (\tau, F(t)) \in D(A^*)\}$$
$$Q_f(t) = \{\tau \in Q(t): \operatorname{div}\tau + f(t) = 0 \text{ in } \Omega\}$$

The stright functional formulation of problem (1.1) follows.

Problem A. It is required to find the functions u and σ, such that

$$u \in L^{\infty}(0, T; D^2(\Omega)); \ u(t) - u_0(t) \in V \tag{1.3}$$

for almost all $t \in [0, T]$.

$$\sigma, \sigma^{\cdot} \in L^{\infty}(0, T; L^2(\Omega)^{n\times n}); \ \sigma(0) = \sigma_0 \tag{1.4}$$
$$\sigma(t) \in Q_f(t) \cap K, \ t \in [0, T]$$
$$\int_{\Omega} \varepsilon(u(t))(\tau - \sigma(t))\,dx - A(\sigma^{\cdot}(t), \tau - \sigma(t)) \leqslant 0 \tag{1.5}$$

for all $\tau \in K$ and almost all $t \in [0, T]$. Here

$$K = \{\tau: \||\tau^D|\|_{L^{\infty}(\Omega)} \leqslant \sqrt{2}k_*\}$$
$$A(\tau, \sigma) = \int_{\Omega} \Big(\frac{1}{n^2K_0}\tau_{jj}\sigma_{ii} + \frac{1}{2\mu}\tau^D\sigma^D\Big)\,dx$$

We assume that the following three conditions hold. The initial stress field is statically possible and admissible, i.e.

$$\sigma_0 \in Q_f(0) \cap K \tag{1.6}$$

A statically possible and safe stress σ_1 exists, and

$$\sigma_1, \sigma_1^{\cdot} \in L^{\infty}(0, T; \Sigma); \ \sigma_1(t) \in Q_f(t), \ t \in [0, T] \tag{1.7}$$
$$\||\sigma_1^D|\|_{L^{\infty}(0, T; L^{\infty}(\Omega))} \leqslant (2(k_*^2 - \delta_*^2))^{1/2}, \quad \delta_* \neq 0$$

A velocity field w_* exists such that

$$w_* \in H^1(\Omega)^n \cap V, \quad \int_{\Omega} \operatorname{div} w_*\,dx = 1 \tag{1.8}$$

The solution of problem A can be reduced to consecutive solution of two problems. In the first problem the stress field is determined, and in the second the velocity field is found. Here are their formulations.

Problem B. It is required to find the stress field σ, satisfying the conditions (1.4) and an inequality of the form

$$\int_{\Omega} \varepsilon(u_0(t))(\tau - \sigma(t))\,dx - A(\sigma^{\cdot}(t), \tau - \sigma(t)) \leqslant 0 \tag{1.9}$$

for all $\tau \in Q_f(t) \cap K$ and almost all $t \in [0, T]$.

Problem C. To find the velocity field u satisfying the condition (1.3), representing for almost all $t \in [0, T]$ a solution of the following variational problem:

$$J_t(u(t)) = a(t) \tag{1.10}$$

Here

$$a(t) = \inf_{v \in W_t} J_t(v), \quad J_t(v) = \sqrt{2} k_* \int_\Omega \left| \varepsilon^D(v) - \frac{1}{2\mu} \sigma^{\cdot D}(t) \right| dx - \int_\gamma F(t) v \, d\Gamma - \int_\Omega f(t) v \, dx$$

$$W_t = \{v \in V + u_0(t): \operatorname{div} v = (nK_0)^{-1} \sigma_{ii}^{\cdot}(t)\}$$

and the tensor σ is a solution of problem B.

2. Problem B. Positive constants c_0 and c_1 exist such, that

$$c_0 \|\sigma\|^2 \leqslant A(\sigma, \sigma), \quad \int_\Omega \sigma\sigma \, dx = \|\sigma\|^2, \quad A(\tau, \sigma) \leqslant c_1 \|\tau\| \|\sigma\| \tag{2.1}$$

Theorem 1. Let conditions (1.2), (1.6), (1.7) hold. Then problem B has a unique solution.

Proof. We consider the following recurrence relation:

$$T_k(\sigma^{k+1}) = \min_{Q_f^{k+1} \cap K} T_k(\tau), \quad \sigma^0 = \sigma_0, \quad k = 0, 1, \ldots, N-1 \tag{2.2}$$

$$T_k(\tau) = \frac{1}{2\Delta} A(\tau - \sigma^k, \tau - \sigma^k) - \int_\Omega \varepsilon(u_0^{k+1}) \tau \, dx$$

$$\Delta = T/N, \quad u_0^k = u_0(k\Delta), \quad Q_f^k = Q_f(k\Delta)$$

The set $Q_f^{k+1} \cap K$ is convex and closed in $L^2(\Omega)^{n \times n}$, therefore the variational problem (2.2) has a unique solution provided that the tensor σ^k is known.

The necessary extremal conditions lead to a series of variational inequalities for determining the tensor

$$\int_\Omega \varepsilon(u_0^{k+1})(\tau - \sigma^{k+1}) \, dx - A\left(\frac{\delta\sigma^k}{\Delta}, \tau - \sigma^{k+1}\right) \leqslant 0, \quad \forall \tau \in Q_f^{k+1} \cap K \tag{2.3}$$

$$\delta\sigma^k = \sigma^{k+1} - \sigma^k, \quad k = 0, 1, \ldots, N-1$$

We introduce the complements in t as follows:

$$Y_N(t) = \sum_{k=0}^{N-1} \sigma^{k+1} \chi_k(t), \quad Y_N^1(t) = \sum_{k=0}^{N-1} \sigma_1^{k+1} \chi_k(t)$$

$$\sigma_N(t) = \sigma_0 + \int_0^t \sum_{k=0}^{N-1} \frac{\delta\sigma^k}{\Delta} \chi_k(\theta) \, d\theta$$

$$\sigma_{1N}(t) = \sigma_1(0) + \int_0^t \sum_{k=0}^{N-1} \frac{\delta\sigma_1^k}{\Delta} \chi_k(\theta) \, d\theta, \quad u_{0N}(t) = \sum_{k=0}^{N-1} u_0^{k+1} \chi_k(t)$$

$$f_N(t) = \sum_{k=0}^{N-1} f^{k+1} \chi_k(t), \quad F_N(t) = \sum_{k=0}^{N-1} F^{k+1} \chi_k(t)$$

where $\chi_k(t)$ is the characteristic function of the semi-interval $[k\Delta, (k+1)\Delta[$ with $k = 0, 1, \ldots, N-2$, and $\chi_{N-1}(t)$ is the characteristic function of the segment $[T - \Delta, T]$.

By virtue of the notation adopted we have

$$\sigma_N(t) - Y_N(t) = (t - (k+1)\Delta)\sigma_N^{\cdot}(t), \quad t \in [k\Delta, (k+1)\Delta[\tag{2.4}$$

Let us choose N so large, that the inequality $\Delta < 1/h$ holds in which

$$h = \frac{M}{\sqrt{2}} \left(\sqrt{k_*^2 - \frac{1}{2}\delta_*^2} - \sqrt{k_*^2 - \delta_*^2} \right)^{-1}$$

$$M = \| |\sigma^{\cdot D}| \|_{L^\infty(0, T; L^\infty(\Omega))}$$

Then

$$\tau^{k+1} = (1 - h\Delta)(\sigma^k - \sigma_1^k) + \sigma_1^{k+1} \in Q_f^{k+1} \cap K$$

Putting in (2.3) $\tau = \tau^{k+1}$, we obtain the inequality

$$A(\sigma_N^{\cdot}(t), \sigma_N^{\cdot}(t)) \leqslant \frac{h}{h\Delta - 1} \left\{ \int_\Omega \varepsilon(u_{0N}(t)) (Y_N(t) - Y_N^1(t)) \, dx - A(\sigma_N^{\cdot}(t), Y_N(t) - Y_N^1(t)) \right\} + \int_\Omega \varepsilon(u_{0N}(t)) (\dot\sigma_{1N}(t) - \dot\sigma_N(t)) \, dx - A(\dot\sigma_N(t), \dot\sigma_{1N}(t)) \tag{2.5}$$

According to conditions (1.2), (1.7) the norms of the functions $\varepsilon(u_{0N})$, σ_{1N}, σ_{1N} are bounded in $L^\infty(0, T; L^2(\Omega)^{n\times n})$. Therefore (2.1) and (2.5) yield the following estimate:

$$\|\sigma_N^{\cdot}(t)\|^2 \leqslant c_2 \int_0^t \|\sigma_N^{\cdot}(\theta)\|^2 d\theta + c_3, \quad t \in [0, T] \tag{2.6}$$

with positive constants c_2 and c_3.

By virtue of estimate (2.6), the sequences $\{\sigma_N\}$ and $\{\sigma_N^{\cdot}\}$ are bounded in $L^\infty(0, T; L^2(\Omega)^{n\times n})$. Choosing, if needed, the subsequences, we find that

$$\begin{aligned} &\sigma_N \to \sigma\ (*) - \quad \text{weakly} \quad L^\infty(0, T; L^2(\Omega)^{n\times n}) \\ &\sigma_N^{\cdot} \to \sigma^{\cdot}\ (*) - \quad \text{weakly} \quad L^\infty(0, T; L^2(\Omega)^{n\times n}) \end{aligned} \tag{2.7}$$

Further, from (2.7) it follows that $\sigma_N(t) \to \sigma(t)$ weakly in $L^2(\Omega)^{n\times n}$ for all $t \in [0, T]$. Thus if the set K is weakly closed in $L^2(\Omega)^{n\times n}$ and $\sigma_N(t) \in K$, then $\sigma(t) \in K$ for all $t \in [0, T]$.

Let φ and w be arbitrary functions in $L^1(0, T)$ and V, respectively. Then

$$\int_0^T \varphi(t) \left\{ \int_\Omega (\sigma_N(t)\, \varepsilon(w) - f_N(t)\, w)\, dx - \int_\gamma F_N(t)\, w\, d\Gamma \right\} dt = 0$$

Using relations (2.4) and (2.7) and passing to the limit in the last identity we find, that $\sigma(t) \in Q_f(t)$ for all $t \in [0, T]$.

Let the tensor function $\varkappa \in C([0, T]; L^2(\Omega)^{n\times n})$ and $\varkappa(t) \in Q_f(t) \cap K$ for $t \in [0, T]$. Taking into account the inequality (2.3), we obtain

$$\int_\Omega \varepsilon(u_{0N}(t))(\varkappa_N(t) - Y_N(t))\, dx - A(\sigma_N^{\cdot}(t), \varkappa_N(t) - Y_N(t)) \leqslant 0$$

$$\varkappa_N(t) = \sum_{k=0}^{N-1} \varkappa((k+1)\Delta)\, \chi_k(t), \quad t \in [0, T]$$

Let us pass to the limit in the last inequality. As a result we obtain the relation

$$\int_0^T I(t, \varkappa(t))\, dt \leqslant 0 \tag{2.8}$$

$$I(t, \varkappa(t)) = \int_\Omega \varepsilon(u_0(t))(\varkappa(t) - \sigma(t))\, dx - A(\sigma^{\cdot}(t), \varkappa(t) - \sigma(t))$$

Let $t_0 \in [0, T]$ be the Lebesgue point of the vector-valued function $t \mapsto \sigma^{\cdot}(t)$ and τ be an arbitrary tensor belonging to $Q_f(t_0) \cap K$. Condition (1.7) ensures the following inclusions:

$$\sigma_\lambda(t) = \lambda(\tau - \sigma_1(t_0)) + \sigma_1(t) \in Q_f(t), \quad t \in [0, T], \quad \lambda \in [0, 1] \tag{2.9}$$

$$\sigma_\lambda \in C([0, T]; \Sigma), \quad \sigma_\lambda(t_0) \in \operatorname{int} K, \quad \lambda \in [0, 1[\tag{2.10}$$

From (2.10) it follows that for $\lambda \in [0, 1[$ a positive number $\delta(\lambda)$, exists such that

$$\sigma_\lambda(t) \in K, \quad t \in [t_0 - \delta(\lambda), t_0 + \delta(\lambda)] \tag{2.11}$$

Let us take an arbitrary function φ from $C([0, T])$, satisfying two conditions:

$$\varphi(t) \in [0, 1], \quad t \in [0, T]; \quad \operatorname{supp} \varphi \subset [t_0 - \delta(\lambda), t_0 + \delta(\lambda)]$$

Then (2.9) and (2.10) yield the inclusion

$$\varkappa(t) = \varphi(t)(\sigma_\lambda(t) - \sigma(t)) + \sigma(t) \in Q_f(t) \cap K, \quad t \in [0, T]$$

Considering inequality (2.8) for the tensor $\varkappa$ constructed above, we obtain

$$\int_{t_0-\delta(\lambda)}^{t_0+\delta(\lambda)} dt\, \varphi(t)\, I(t, \sigma_\lambda(t)) \leqslant 0$$

By virtue of the choice of the point t_0 and the arbitrariness of φ, the last inequality yields the relation $I(t_0, \sigma_\lambda(t_0)) \leqslant 0$. Letting λ tend to unity, we arrive at inequality (1.9) when $t = t_0$. Since the set of points t_0 is a set of the total measure in the interval $[0, T]$, it follows that inequality (1.9) holds for almost all t belonging to $[0, T]$.

The proof of the uniqueness of the solution of problem B is standard (see e.g. /1/).

3. The connection between problem A and problem B and C. We shall show that the set on which the solution of problem C is sought, is non-empty.

Let us introduce the subspaces L_0^2 and $H_0^1(\Omega)^n$ as follows:

$$L_0^2 = \left\{ f \in L^2(\Omega) : \int_\Omega f\, dx = 0 \right\}$$

$$H_0^1(\Omega)^n = \{ u \in H^1(\Omega)^n : u = 0 \text{ on } \Gamma \}$$

As was shown in /14/, for any function f belonging to L_0^2 there exists a vector function u belonging to $H_0^1(\Omega)^n$ such that

$$\operatorname{div} u = f \text{ in } \Omega; \ \|| u \|| \leqslant c_4 \| f \|_{L^2(\Omega)} \tag{3.1}$$

and the positive constant is independent of f.

Consider the following variational problem:

$$\|| u_f \|| = \min_{u \in H_0^1(\Omega)^n, \ \operatorname{div} u = f} \|| u \|| \tag{3.2}$$

Problem (3.2) has a unique solution for any f from L_0^2; consequently it determines the operator π: $L_0^2 \to H_0^1(\Omega)^n$ such that $\pi f = u_f$. The operator is linear and continuous by virtue of (3.1).

Lemma 1. Let the tensor σ be a solution of problem B. Then a function u_* exists such, that

$$u_* \in L^\infty(0, T; H^1(\Omega)^n);$$
$$u_*(t) - u_0(t) \in V, \ \operatorname{div} u_*(t) = (nK_0)^{-1}\sigma_{ii}^{\cdot}(t) \text{ in } \Omega$$

Indeed, by virtue of the definition of the operator π and condition (1.8), we can take as u_* a function of the form

$$u_*(t) = u_0(t) + v_*(t) + w_* \int_\Omega \beta(t)\, dx$$
$$\beta(t) = (nK_0)^{-1}\sigma_{ii}^{\cdot}(t) - \operatorname{div} u_0(t), \quad v_*(t) = \pi\left(\beta(t) - \operatorname{div} w_* \int_\Omega \beta(t)\, dx\right)$$

Theorem 2. The pair of functions σ and u represent a solution of problem A if and only if the functions are solutions of problems B and C, respectively.

The proof of Theorem 2 is preceded by a lemma.

Lemma 2. Let the tensor σ be a solution of problem B. Then the following relation holds for almost all $t \in [0, T]$:

$$a(t) = \int_\Omega (\varepsilon(u_0(t))\sigma(t) - f(t)u_0(t))\, dx - \int_\gamma F(t)u_0(t)\, d\Gamma - A(\sigma^{\cdot}(t), \sigma(t)) \tag{3.3}$$

Proof. We introduce a dual pair of Banach spaces

$$P^* = \{p^* = (\tau, g): \ \|p^*\|_* = \|\tau\|_\Sigma + \||g|\|_{L^\infty(\gamma)} < +\infty\}$$
$$P = \{p = (\varkappa, s): \varkappa = (\varkappa_{ij}), \ \varkappa_{ij} = \varkappa_{ji} \in L^1(\Omega), \ \varkappa_{ii} \in L^2(\Omega)$$
$$i, j = 1, 2, \ldots, n; \ s \in L^1(\gamma)^n\}$$
$$\langle p^*, p\rangle = \int_\Omega \tau\varkappa\, dx + \int_\gamma gs\, d\Gamma, \quad p^* = (\tau, g), \quad p = (\varkappa, s)$$

We also define a linear continuous operator A: $V \to P$ and a Lagrangian l

$$Av = (\varepsilon(v), \ -v|_\gamma), \quad v \in V\}$$
$$l_t(v, p^*) = \langle p^*, Av\rangle + \int_\Omega (\varepsilon(u_*(t))\tau - f(t)v - \varepsilon(u_*(t) -$$
$$u_0(t))\sigma(t))\, dx - A(\sigma^{\cdot}(t), \tau) - G_{1t}^*(p^*), \quad v \in V, \quad p^* = (\tau, g)$$
$$G_{1t}^*(p^*) = \begin{cases} 0, & \text{if } p^* = (\tau, F(t)), \quad \tau \in K \\ +\infty & \text{otherwise} \end{cases}$$

Let us write the relations connecting the functionals of the dual problems with the Lagrangian l_t

$$\sup_{q^* \in P^*} l_t(v, q^*) = \Phi_t(v), \quad v \in V \tag{3.4}$$
$$\Phi_t(v) = G_t(Av) - \int_\Omega f(t)\, v\, dx$$
$$\inf_{v \in V} l_t(v, q^*) = R_t(q^*), \quad q^* \in P^* \tag{3.5}$$

We shall show that the following relations hold for almost all $t \in [0, T]$:

$$\max_{q^*} R_t(q^*) = R_t(p^*(t)), \ p^*(t) = (\sigma(t), F(t)) \tag{3.6}$$
$$R_t(p^*(t)) = \inf_V \Phi(v) \tag{3.7}$$

Indeed, from (3.5) we obtain the expression for the functional R_t

$$R_t(q^*) = \int_\Omega \varepsilon(u_0(t))\tau\, dx - A(\sigma^{\cdot}(t), \tau), \quad \text{if } q^* = (\tau, F(t)) \tag{3.8}$$
$$\tau \in Q_t(t) \cap K, \ R_t(q^*) = -\infty \text{ otherwise}$$

Hence, Eq. (3.6) is a direct consequence of inequality (1.9). Consider the perturbed problem

$$h_t(p) = \inf_V \left\{G_t(Av+p) - \int_\Omega f(t)\, v\,dx\right\}$$

Eq.(3.7) holds if the function $h_t(p)$ satisfies the following two conditions (/15/, ch.3, proposition 2.1) for almost all $t \in [0, T]$: 1) $h_t(0)$ is a finite quantity, 2) the function $p \to h_t(p)$ is semicontinuous from below at the zero of the space P.

Computing the exact upper bound in relation (3.4), we arrive at the formula

$$G_t(Av+p) = \sqrt{2}\, k_* \int_\Omega \left| \varepsilon^D(v + u_*(t)) + \varkappa^D - \frac{1}{2\mu}\sigma^{\cdot D}(t)\right| dx -$$
$$\int_\Omega \varepsilon(u_*(t) - u_0(t))\,\sigma(t)\,dx + \int_\gamma F(t)(s-v)\,d\Gamma +$$
$$\begin{cases} 0, & \text{if } \varkappa_{ii} + \operatorname{div} v = 0 \text{ in } \Omega \\ +\infty & \text{otherwise случае} \end{cases}, \quad v \in V, \quad p = (\varkappa, S)$$

from which we obtain the relation

$$a(t) = h_t(0) - \int_\Omega f(t)\,u_0(t)\,dx - \int_\gamma F(t)\,u_0(t)\,d\Gamma \tag{3.9}$$

An upper estimate for $h_t(0)$ can be obtained from (3.9), since $a(t) \leqslant J_t(u_*(t))$. To obtain a lower estimate, we use condition (1.7). As a result we obtain

$$J_t(v + u_*(t)) \geqslant \sqrt{2}\,(k_* - \sqrt{k_*^2 - \delta_*^2}) \int_\Omega |\varepsilon^D(v)|\,dx - \tag{3.10}$$
$$\int_\Omega \left(f(t)\,u_*(t) + \sqrt{2}\,k_* \left|\varepsilon^D(u_*(t)) - \frac{1}{2\mu}\sigma^{\cdot D}(t)\right|\right)dx - \int_\gamma F(t)\,u_*(t)\,d\Gamma, \quad v \in V$$

The required lower estimate then follows from (3.9) and (3.10).

To confirm condition 2), we will choose the function v_* as follows:

$$v_* = \pi\left(\operatorname{div} w_* \int_\Omega \varkappa_{ii}\,dx - \varkappa_{ii}\right) - w_* \int_\Omega \varkappa_{ii}\,dx$$

According to the definition of the operator π we have

$$v_* \in V, \ \operatorname{div} v_* + \varkappa_{ii} = 0 \text{ in } \Omega, \quad |||v_*||| \leqslant c_5 \|\varkappa_{ii}\|_{L^2(\Omega)} \tag{3.11}$$

where the constant c_5 is independent of $p = (\varkappa, s)$.

Let us put $v = w + v_*$ in the formula for $G_t(Av+p)$ and obtain a lower estimate the function $h_t(p)$ as follows:

$$h_t(p) = \inf_{w \in V,\ \operatorname{div} w = 0} \left\{G_t(A(w+v_*)+p) - \int_\Omega f(t)(w+v_*)\,dx\right\} \geqslant$$
$$h_t(0) - \int_\Omega (\sqrt{2}\,k_* |\varepsilon^D(v_*) + \varkappa^D| + f(t)\,v_*)\,dx + \int_\gamma F(t)(s - v_*)\,d\Gamma$$

This, togehter with (1.2) and (3.11), yields the relation

$$h_t(p) \geqslant h_t(0) - c_6\|p\|_P, \quad p \in P$$

with a positive constant c_6 independent of p. Therefore condition 2) holds. The assertion of the lemma can now be derived from (3.9), (3.8), (3.6), (3.7).

Theorem 2 follows directly from Lemma 2, if we note that in all cases the pair of functions $v(t) = u(t) - u_*(t)$ and $p^*(t) = (\sigma(t), F(t))$ represents a saddle point of the Lagrangian l_t on the set $V \times P^*$ for almost all $t \in [0, T]$.

Theorems 1 and 2 show that the solvability of problem A is equivalent to the solvability of problem C.

Next we give an example showing that problem A and C may have no solutions. Let us consider a plane problem for a concentric ring. Passing to the polar ρ, θ coordinate system with a pole at the centre of the ring, we specify the load conditions as follows:

$$f = 0, \ \sigma_0 = 0 \text{ in } \Omega; \ \gamma = \varnothing; \ u = 0 \text{ when } \rho = R_1$$
$$u = (0, U_\theta) \text{ when } \rho = R_2, \ U_\theta = \text{const}$$

where R_1 and R_2 are the radii of the inner and outer ring contours, respectively.

Direct subsitution shows that the unique solution of problem B is represented by a tensor σ of the form

$$\sigma = \begin{Vmatrix} 0 & \sigma_{\rho\theta} \\ \sigma_{\rho\theta} & 0 \end{Vmatrix}, \quad \sigma_{\rho\theta} = k_* \left(\frac{R_1}{\rho}\right)^2 \begin{cases} tU_\theta/U_*, & t \in [0, t_0] \\ 1, & t \in [t_0, T] \end{cases}$$
$$t_0 = \frac{U_*}{U_\theta}, \quad U_* = \frac{\alpha^2 - 1}{\alpha^2}\,\frac{k_*}{2\mu} R_2, \quad \alpha = \frac{R_2}{R_1}$$

From (1.5) by necessity it follows that

$$\varepsilon(u(t)) = 0, \ \text{if} \quad R_1 < \rho < R_2, \quad t_0 < t < T$$

However, the last equation has no solutions in the class $D^2(\Omega)$ satisfying the boundary conditions.

4. Augmented functional formulation of the classical problem. Let us introduce an additional space of vector functions

$$V_+ = \left\{ v \in L^{n/(n-1)}(\Omega)^n : \| v \|_+ = \sup_{\substack{\|p^*\|_* \leqslant 1, \\ p^* = (\tau, g) \in D(A^*)}} \int_\Omega v \operatorname{div} \tau \, dx < +\infty \right\}$$

and give an augmented formulation of the classical problem.

Problem A^+. It is required to find a vector function u and tensor function σ, satisfying conditions (1.4) and such, that

$$u \in L^\infty(0, T; L^{n(n-1)}(\Omega)^n) \tag{4.1}$$

$$u(t) \in V_+ + u_0(t) \text{ for almost all } t \in [0, T]$$

$$\int_\Omega \{\varepsilon(u_0(t))(\tau - \sigma(t)) - (u(t) - u_0(t)) \operatorname{div}(\tau - \sigma(t))\} \, dx - \tag{4.2}$$

$$A(\sigma^{\cdot}(t), \tau - \sigma(t)) \leqslant 0, \quad \forall \tau \in Q(t) \cap K$$

for almost all $t \in [0, T]$.

We note that if the pair of functions σ and u is a solution of problem A^+, then the tensor function σ is a solution of problem B.

The reasons for introducing such a formulation of the problem are as follows. Let us consider the auxilliary problem C^+.

Problem C^+. It is required to find the velocity field u satisfying conditions (4.1) and such, that

$$I_t(u(t)) = \min_{v \in W_t^+} I_t(v), \quad t \in [0, T] \tag{4.3}$$

$$W_t^+ = \{v \in V_+ + u_0(t): \operatorname{div} v = (nK_0)^{-1} \sigma_{ii}(t)\}$$

$$I_t(v) = \sup_{\tau \in Q(t) \cap K} L_t(v, \tau), \quad L_t(v, \tau) =$$

$$\int_\Omega \{\varepsilon(u_0(t)) \tau - (v - u_0(t)) \operatorname{div} \tau\} \, dx - A(\sigma^{\cdot}(t), \tau) - \int_\Omega f(t) v \, dx - \int_\gamma F(t) u_0(t) \, d\Gamma$$

and the tensor σ is a solution of problem B.

The theorem which follows shows that problem C^+ represents a variational extension of problem C.

Theorem 3. The following relations hold for almost all $t \in [0, T]$:

$$I_t(v) = J_t(v), \; v \in W_t \tag{4.4}$$

$$\min_{v \in W_t^+} I_t(v) = \inf_{v \in W_t} J_t(v) \tag{4.5}$$

We recall that replacing **inf** by **min** means that the corresponding variational problem has a solution.

The following assertion is an analogue of Theorem 2.

Theorem 4. The pair of functions σ and u is a solution of problem A^+ if and only if the functions are solutions of problems B and C respectively.

By virtue of Theorems 1, 3 and 4 problems A^+ always has a solution, and the set of its solutions is convex. Moreover, all elements of this set have the same tensor σ.

Proof of Theorem 3. By virtue of the definition of the set $Q(t)$, the following relation holds for any vector function $v \in V + u_0(t)$ and any tensor function $\tau \in Q(t)$:

$$L_t(v, \tau) = \int_\Omega (\varepsilon(v) \tau - f(t) v) \, dx - \int_\gamma F(t) v \, d\Gamma - A(\sigma^{\cdot}(t), \tau) \tag{4.6}$$

Therefore

$$J_t(v) = \sup_{\tau \in K} L_t(v, \tau) \geqslant \sup_{\tau \in Q(t) \cap K} L_t(v, \tau) = I_t(v), \quad \forall v \in W_t.$$

To prove the converse inequality we take any smooth function $\tau \in K$ and a sequence of the infinitely differentiable functions φ_m, which have a compact carrier in Ω and satisfy the following conditions:

$$\varphi_m(x) \in [0, 1], \; x \in \Omega; \; \varphi_m(x) \to 1 \text{ for almost all } x \in \Omega$$

It can be shown that $\sigma_{1kk} \in L^n(\Omega)$ /9, 10/; therefore we have

$$\tau_m(t) = \varphi_m(\tau - \sigma_1(t)) + \sigma_1(t) \in Q(t) \cap K, \quad t \in [0, T]$$

Let us put $\tau = \tau_m$ (4.6) and pass to the limit. By virtue of the Lebesgue theorem and the definition of the functional I_t we obtain the inequality

$$I_t(v) \geqslant \int_\Omega (\varepsilon(v)\tau - f(t)v)\,dx - \int_N F(t)\,v\,d\Gamma - A(\sigma^\cdot(t), \tau), \quad v \in W_t$$

for all smooth tensors τ in K. This implies the converse inequality and the assertion (4.4) of the theorem follows from it. Before proving assertion (4.5) we state two lemmas.

Lemma 3. The following relation holds:

$$\inf_{W(0,T)} \int_0^T (J_t(v(t)) - a(t))^2\,dt = 0 \tag{4.7}$$

$$W = (0, T) = \{v \in L^\infty(0, T; D^2(\Omega)) : v(t) \in W_t, \quad t \in [0, T]\}$$

Lemma 4. A sequence $\{u_m\} \in W(0, T)$, and functions $u \in L^\infty(0, T; L^{n/(n-1)}(\Omega)^n)$ and $\alpha \in L^\infty(0, T)$ exist such, that the following assertions hold for almost all $t \in [0, T]$:

$$J_t(u_m(t)) \to a(t) \tag{4.8}$$

$$u_m(t) \to u(t) \text{ strongly in } L^{n/(n-1)}(\Omega)^n \tag{4.9}$$

$$\operatorname{div} u(t) = (nK_0)^{-1}\sigma_{ii}^\cdot(t) \quad \text{in } \Omega \tag{4.10}$$

$$\| u(t) - u_0(t) \|_+ \leqslant \alpha(t) \tag{4.11}$$

The proof of Lemma 3 utilizing the convexity of the functional J_t and the definition of the function a is standard and will not be given here.

Proof of Lemma 4. Conditions (1.2), Theorem 1, Lemma 1 and inequality (3.10) together yield the estimate

$$J_t(w(t)) + \varphi(t) \geqslant \sqrt{2}\,(k_* - \sqrt{k_*^2 - \delta_*^2}) \int_\Omega |\varepsilon(w(t) - u_*(t))|\,dx, \ \forall w \in W(0, T), \quad t \in [0, T] \tag{4.12}$$

with a certain function φ from $L^\infty(0, T)$.

Taking into account the continuity of the embedding of the space $D^2(\Omega)$ into the spaces $L^{n/(n-1)}(\Omega)^n$ and $L^1(\Gamma)^n$, we obtain the following relations from inequality (4.12):

$$\| w(t) - u_*(t) \|_{L^{n/(n-1)}(\Omega)^n} \leqslant c_7 (J_t(w(t)) + \varphi(t)) \tag{4.13}$$

$$\| A(w(t) - u^*(t) \|_P = \sup_{\|p^*\|_* \leqslant 1} < p^*, \ A(w(t) - u_*(t)) > \leqslant c_8 (J_t(w(t)) + \varphi(t)) \tag{4.14}$$

which hold for almost all $t \in [0, T]$ and for all $w \in W(0, T)$.

Let $\{w_m\}$ be the minimizing sequence of problem (4.7). Then

$$\int_0^T (J_t(w_m(t)) - a(t))^2\,dt \to 0 \tag{4.15}$$

Since $a \in L^\infty(0, T)$, then from (4.13), (4.15) there follows the boundedness of the sequence $\{w_m\}$ in $L^2(0, T; L^{n/(n-1)}(\Omega)^n)$. Omitting the standard considerations, we assert that a sequence $\{u_m\}$ exists belonging to the convex shell of the set $\{w_m'\}$, for which assertion (4.9) of the lemma holds, with some function u from $L^2(0, T; L^{n/(n-1)}(\Omega)^n)$.

It is clear that the sequence $\{u_m\}$ will be the minimizing one in problem (4.7), and this proves assertions (4.8), (4.10).

Assuming $w = u_m$ in inequality (4.13) and passing to the limit, taking relations (4.8), (4.9) into account, we find that $u \in L^\infty(0, T; L^{n/(n-1)}(\Omega)^n)$.

Further, using the definitions of the set $D(A^*)$, the norm $\| \cdot \|_+$ and relation (4.14), we obtain the following inequality for $w = u_m$:

$$\sup_{\substack{\|p^*\|_* \leqslant 1,\\ p^* = (\tau, g) \in D(A^*)}} \int_\Omega (u_m(t) - u_*(t)) \operatorname{div} \tau\,dx \leqslant c_8 (J_t(u_m(t)) + \varphi(t))$$

for which we obtain, passing to the limit, the estimate

$$\| u(t) - u_*(t) \|_+ \leqslant c_8 (a(t) + \varphi(t)), \quad t \in [0, T]$$

In this case the function $\alpha(t)$ sought will be equal to the sum of the left and right sides of the last inequality, when $u(t) = u_0(t)$. The lemma is proved.

Let us now prove assertion (4.5). From (4.4) it follows that

$$J_t(u_m(t)) = I_t(u_m(t)) \geqslant L_t(u_m(t), \tau), \ \tau \in Q(t) \cap K$$

Passing to the limit and taking (4.8), (4.9) into account, we arrive at the inequality

$$L_t(u(t), \tau) \leqslant a(t), \quad \tau \in Q(t) \cap K, \quad t \in [0, T]$$

Taking into account the structure of the Lagrangian L_t and relation (3.3), we obtain

$$a(t) = L_t(v, \sigma(t)), \quad \forall v \in W_t^+$$

The last two relations imply that the pair of functions $u(t)$ and $\sigma(t)$ represent a saddle point of the Lagrangian L_t on the set $W_t^+ \times (Q(t) \cap K)$ for almost all $t \in [0, T]$. Arguments used normally in the theory of duality, lead to the required results, thus proving Theorem 3.

Proof of Theorem 4. Let the pair σ and u be a solution of problem A^+. Then the tensor σ will be a solution of problem B. Assuming $\tau = \varphi E + \sigma(t)$ in (4.2) where φ is an arbitrary smooth function equal to zero near the boundary, we obtain

$$\operatorname{div} u(t) = (nK_0)^{-1}\sigma_{ii}{}^{\cdot}(t), \quad t \in [0, T]$$

Therefore $u(t) \in W_t^+$ for almost all $t \in [0, T]$.

The following dual inequality follows from (4.2):

$$L_t(u(t), \tau) \leqslant a(t) \leqslant L_t(v, \sigma(t)), \quad v \in W_t^+, \quad \tau \in Q(t) \cap K \tag{4.16}$$

and this means that the function u is a solution of problme C^+.

Conversely, if the functions σ and u are solutions of problems B and C^+, then from Theorem 1, Eqs.(3.3), (4.5) and the definition of the Lagrangian L_t the inequalities (4.16) follow. Thus the pair σ and u represents a solution of problem A^+. The theorem is proved.

Returning now to the problem formulated at the end of Sect.3, we note that the velocity field in the augmented formulation is found from the inequality (4.2). Omitting the intermediate manipulations, we write the final solution as follows:

$$u = (0, u_\theta)$$

$$u_\theta = \frac{U_\theta}{\alpha^2 - 1}\left(\alpha^2 \frac{\rho}{R_2} - \frac{R_2}{\rho}\right), \quad t \in [0, t_0]; \quad u_\theta = \frac{U_\theta}{R_2}\rho, \quad t \in [t_0, T]$$

The solution obtained has a discontinuity on the inner contour of the ring when $t > t_0$. It can be shown that in this case problem A^+ has a unique solution.

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